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AN EQUIVARIANT DISCRETE MODEL FOR COMPLEXIFIED ARRANGEMENT COMPLEMENTS

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Abstract. We define a partial ordering on the set $\mathcal{Q} = \mathcal{Q}(\mathsf{M})$ of pairs of topes of an oriented matroid M, and show the geometric realization $|\mathcal{Q}|$ of the order complex of $\mathcal{Q}$ has the same homotopy type as the Salvetti complex of M. For any element e of the ground set, the complex $|\mathcal{Q}_e|$ associated to the rank-one oriented matroid on $\{e\}$ has the homotopy type of the circle. There is a natural free simplicial action of $\mathbb{Z}_4$ on $|\mathcal{Q}|$, with orbit space isomorphic to the order complex of the poset $\mathcal{Q}(\mathsf{M}, e)$ associated to the pointed (or affine) oriented matroid (M, e). If M is the oriented matroid of an arrangement $\mathcal{A}$ of linear hyperplanes in $\mathbb{R}^n$, the $\mathbb{Z}_4$ action corresponds to the diagonal action of $\mathbb{C}^*$ on the complement M of the complexification of $\mathcal{A}$: $|\mathcal{Q}|$ is equivariantly homotopy-equivalent to M under the identification of $\mathbb{Z}_4$ with the multiplicative subgroup $\{\pm 1, \pm i\} \subset \mathbb{C}^*$, and $|\mathcal{Q}(\mathsf{M}, e)|$ is homotopy-equivalent to the complement of the decone of $\mathcal{A}$ relative to the hyperplane corresponding to e. All constructions and arguments are carried out at the level of the underlying posets.

We also show that the class of fundamental groups of such complexes is strictly larger than the class of fundamental groups of complements of complex hyperplane arrangements. Specifically, the group of the non-Pappus arrangement is not isomorphic to any realizable arrangement group. The argument uses new structural results concerning the degree-one resonance varieties of small matroids.

1. Introduction

An arrangement of hyperplanes is a set $\mathcal{A} = \{H_1, \ldots, H_n\}$ of linear or affine codimension-one subspaces of $\mathbb{C}^d$. An arrangement is *complexified* if each H_i has a defining equation with real coefficients; in this case the underlying real arrangement $\{H_1 \cap \mathbb{R}^d, \ldots, H_n \cap \mathbb{R}^d\}$ is denoted $\mathcal{A}_\mathbb{R}$. A main topic in the theory of hyperplane arrangements is the study of combinatorial invariants of the topology of the complement $M(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup \mathcal{A}$.

The arrangement $\mathcal{A}$ is called *central* if all its hyperplanes contain the origin; in this case, $M(\mathcal{A})$ carries the natural (diagonal) $\mathbb{C}^*$-action. One of the many consequences of this fact is the following topological property. Fix an element $H_0 \in \mathcal{A}$ and let H_0' be a parallel translate of H_0 that does not contain the origin. Let $d\mathcal{A}$ be the *decone* of $\mathcal{A}$ relative to H_0, the arrangement $\{H \cap H_0' \mid H \in \mathcal{A} \setminus H_0\}$ in $H_0' \cong \mathbb{C}^{d-1}$. Then there is a diffeomorphism

$$M(\mathcal{A}) \cong \mathbb{C}^* \times M(d\mathcal{A}).$$

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There exist combinatorially defined complexes that model the homotopy type of $\mathsf{M}(\mathcal{A})$, e.g., by work of Salvetti [20] in the complexified case, and Björner and Ziegler [5] in the general case. These complexes are finite, therefore cannot model the circle action of $\mathsf{S}^1 \subset \mathbb{C}^*$ on $\mathsf{M}(\mathcal{A})$.

In principle, there are two ways out of this situation: either to develop 'continuous' combinatorial models that can carry a circle action, or to let a 'discretized' S^1 act on the known combinatorial models. A continuous approach has been attempted, e.g., in [2], and is as yet not fully developed. Here we explore the second possibility, also in view of the fact that the simplicial complexes mentioned above are defined in the general setting of *pseudo-sphere arrangements*, where no original linear space with $\mathbb{C}^*$-action exists.

The known discrete complexes depend only on the combinatorics of arrangements of real codimension-one pseudo-spheres in S^{d-1}, encoded by the associated oriented matroid or 2-matroid, respectively, and are defined as the *order complexes* of certain partially-ordered sets, or posets. The order complex of a poset $\mathcal{P}$ is the abstract simplicial complex $\Delta(\mathcal{P})$ whose simplices are the linearly-ordered subsets, or *chains*, of $\mathcal{P}$. Order-preserving and order-reversing maps of posets induce simplicial maps of order complexes. The geometric realization of $\Delta(\mathcal{P})$ is denoted $|\mathcal{P}|$, and is called the *geometric realization* of $\mathcal{P}$ (see Remark 2.3).

Here we treat only complexified arrangements, in the general setting of oriented matroids. Associated to a loop-free oriented matroid M, one has the *Salvetti poset* $\mathcal{S} = \mathcal{S}(\mathsf{M})$ whose geometric realization $|\mathcal{S}|$ has the homotopy type of $\mathsf{M}(\mathcal{A})$ in case M is realized by the real arrangement $\mathcal{A}_{\mathbb{R}}$. In general, by a result of Deshpande [9], $|\mathcal{S}|$ has the homotopy type of the tangent bundle complement of the arrangement of pseudo-spheres associated to M (see Definition 3.1). If e_0 is a fixed element of the ground set of M (corresponding to $\mathsf{H}_0 \in \mathcal{A}$) one has the pointed (or affine) oriented matroid (M, e_0), and an associated subposet $d\mathcal{S} = \mathcal{S}(\mathsf{M}, e_0)$ of $\mathcal{S}$, with $|d\mathcal{S}|$ homotopy equivalent to the complement of the decone $d\mathcal{A}$ of $\mathcal{A}$ relative to H_0.

In this paper, after a preparatory section on the basics of poset topology, we

- define posets $\mathcal{Q} = \mathcal{Q}(\mathsf{M})$ and $d\mathcal{Q} = \mathcal{Q}(\mathsf{M}, e_0) \subseteq \mathcal{Q}$ and an order-preserving map $\mathcal{S} \to \mathcal{Q}$ inducing homotopy equivalences $|\mathcal{S}| \simeq |\mathcal{Q}|$ and $|d\mathcal{S}| \simeq |d\mathcal{Q}|$;
- define a natural free action of $\mathbb{Z}_4$ on $\mathcal{Q}$ by order-reversing and -preserving bijections;
- define an equivariant order-preserving map $\mathcal{Q}_{e_0} \times d\mathcal{Q} \to \mathcal{Q}$, where $\mathcal{Q}_{e_0}$ is the poset associated with $\mathsf{M}|_{\{e_0\}}$ and $\mathbb{Z}_4$ acts trivially on $d\mathcal{Q}$, inducing a homotopy equivalence $|\mathcal{Q}_{e_0}| \times |d\mathcal{Q}| \simeq |\mathcal{Q}|$. Then $|\mathcal{S}_{e_0}| \times |d\mathcal{S}| \simeq |\mathcal{S}|$ as well.

Thus we obtain a combinatorial version of the cone-decone property of complexified hyperplane arrangements, which holds in the ostensibly more general setting of oriented matroids, realizable or not. As a corollary we obtain the main result of [8], a product decomposition $\pi_1(|\mathcal{S}|) \cong \mathbb{Z} \times \pi_1(|d\mathcal{S}|)$ of fundamental groups, originally proved via complicated manipulation of group presentations. Our work also partly answers a question of Ziegler [22, Problem 7.7].

Finally we show that this setting is indeed more general, by displaying an oriented matroid M, an orientation of the non-Pappus matroid, for which $\pi_1(|\mathcal{Q}|)$ is not isomorphic to the fundamental group of the complement of any complex hyperplane arrangement. To our knowledge no such example has appeared in the literature. The argument uses properties of degree-one resonance varieties of small matroids.

2. Poset topology

Definition 2.1. A partially-ordered set (or *poset*) is a pair $(\mathcal{P}, \leq)$ where $\mathcal{P}$ is a set and $\leq$ a partial order relation on $\mathcal{P}$. A morphism of posets $(\mathcal{P}, \leq_{\mathcal{P}}) \to (\mathcal{Q}, \leq_{\mathcal{Q}})$ is an order-preserving function $f : \mathcal{P} \to \mathcal{Q}$, i.e., one for which $f(p_1) \leq_{\mathcal{Q}} f(p_2)$ whenever $p_1 \leq_{\mathcal{P}} p_2$; it is an isomorphism if f is bijective, and in this case we will write $(\mathcal{P}, \leq_{\mathcal{P}}) \cong (\mathcal{Q} \leq_{\mathcal{Q}})$. We will write Pos for the category of posets and order-preserving functions. A *chain* in the poset $(\mathcal{P}, \leq)$ is a subset of $\mathcal{P}$ that is totally ordered by $\leq_{\mathcal{P}}$. The *product* of two posets $(\mathcal{P}, \leq_{\mathcal{P}})$ and $(\mathcal{Q}, \leq_{\mathcal{Q}})$ is $(\mathcal{P} \times \mathcal{Q}, \leq_{\mathcal{P}\times\mathcal{Q}})$, where $(p_1, q_1) \leq_{(\mathcal{P}\times\mathcal{Q})} (p_2, q_2)$ if and only if $p_1 \leq_{\mathcal{P}} p_2$ and $q_1 \leq_{\mathcal{Q}} q_2$.

The *opposite* or 'order dual' of a given poset $(\mathcal{P}, \leq_{\mathcal{P}})$ is the poset $(\mathcal{P}, \leq_{\mathcal{P}})^{\mathrm{op}} = (\mathcal{P}, \leq_{\mathcal{P}}^{\mathrm{op}})$ where $p_1 \leq_{\mathcal{P}}^{\mathrm{op}} p_2$ if and only if $p_2 \leq_{\mathcal{P}} p_1$.

Remark 2.2 (Notation). It is customary to denote a poset $(\mathcal{P}, \leq)$ by its underlying set $\mathcal{P}$ when the order relation is understood.

Let $\mathcal{P}$ be a poset. Let $(\Delta(\mathcal{P}), \leq)$ be the poset of chains in $\mathcal{P}$, with $\sigma \leq \tau$ if and only if $\sigma \subseteq \tau$. The poset $\Delta(\mathcal{P})$ is an abstract simplicial complex with vertex set $\mathcal{P}$, called the *order complex* of $\mathcal{P}$. The standard geometric realization of $\Delta(\mathcal{P})$ will be denoted by $|\mathcal{P}|$, and called the *geometric realization* of $\mathcal{P}$. We refer to [15] as a general reference for poset topology.

Remark 2.3. The terminology leads to no conflict: if $\mathcal{P}$ is a simplicial complex, there is a simplicial homeomorphism of $|\Delta(\mathcal{P})|$ to the barycentric subdivision of $|\mathcal{P}|$. See also Remark 2.10 below.

As is customary, we refer to the homotopy type of $|\mathcal{P}|$ when speaking of "the homotopy type of the poset $\mathcal{P}$". In particular, we will say that posets $\mathcal{P}$ and $\mathcal{Q}$ are homotopy equivalent (written $\mathcal{P} \simeq \mathcal{Q}$) if $|\mathcal{P}|$ and $|\mathcal{Q}|$ are.

Remark 2.4.

(a) For every poset $\mathcal{P}$ we have $\Delta(\mathcal{P}) = \Delta(\mathcal{P}^{\mathrm{op}})$.
(b) If $\mathcal{P}$ and $\mathcal{Q}$ are posets, then $|\mathcal{P} \times \mathcal{Q}|$ is homeomorphic to $|\mathcal{P}| \times |\mathcal{Q}|$. (In fact $\Delta(\mathcal{P} \times \mathcal{Q})$ is a triangulation of $|\mathcal{P}| \times |\mathcal{Q}|$.) See [15, Theorem 10.21] for a generalization.

The following "Quillen Lemma" is widely used.

Lemma 2.5 ([19])**.** *Let $f : \mathcal{P} \to \mathcal{Q}$ be a poset map. If $f^{-1}(\mathcal{Q}_{\geq q})$ is contractible for all $q \in \mathcal{Q}$, then $\mathcal{P} \simeq \mathcal{Q}$.*

Remark 2.6. The condition of Lemma 2.5 can be replaced by "$f^{-1}(\mathcal{Q}_{\leq q})$ is contractible for all $q \in \mathcal{Q}$" via Remark 2.4(a).

Definition 2.7. An order-preserving function $f : \mathcal{P} \to \mathcal{P}$ is *monotone* if either $f(p) \geq p$ for all $p \in \mathcal{P}$ or $f(p) \leq p$ for all $p \in \mathcal{P}$.

Lemma 2.8 (Theorem 13.22(b) in [15])**.** *Let $f : \mathcal{P} \to \mathcal{P}$ be a monotone poset map. Then $\mathcal{P} \simeq \mathsf{fix}(f)$.*

Remark 2.9. If a poset $\mathcal{P}$ has a unique maximal element p, then $\mathcal{P}$ is contractible because its order complex is the cone over the order complex of $\mathcal{P} \setminus \{p\}$.

Remark 2.10. For every poset $\mathcal{P}$, there is a canonical homotopy equivalence $\Delta(\mathcal{P}) \simeq \mathcal{P}$ (e.g., by the function $\Delta(\mathcal{P}) \to \mathcal{P}$, $\omega \mapsto \min \omega$).

3. Discrete circle action on complexified arrangements

For the remainder of this paper fix a rank r oriented matroid on finite ground set E and let $\mathcal{F}$ be its set of covectors. For an introduction to the theory of oriented matroids see [4]: here we recall only what is needed in the following.

Definition 3.1 ([4, Definition 5.1.3])**.** A *rank* r *arrangement of pseudo-spheres* is a set $\mathcal{A} = \{S_e\}_{e\in E}$ of centrally symmetric PL-homeomorphic embeddings of S^{r-2} in S^{r-1} satisfying $\bigcap \mathcal{A} = \emptyset$ and, for all $\mathcal{B} \subseteq \mathcal{A}$, $\bigcap \mathcal{B}$ is a PL-sphere, together with a choice of a connected component S_e^+ of $S^r \setminus S_e$ for every $e \in E$.

The set of *real signs* is $\{+, 0, -\}$, and the map

$$\sigma_{\mathcal{A}} : S^{r-1} \to \{+, 0, -\}^E; \quad \sigma_{\mathcal{A}}(x)_e := \begin{cases} + & \text{if } x \in S_e^+, \\ 0 & \text{if } x \in S_e, \\ - & \text{else,} \end{cases}$$

associates a *sign vector* to every point of the sphere. Notice that the zero vector $\hat{0} := (0, \dots, 0)$ is not in the image of $\sigma_{\mathcal{A}}$.

The set of covectors of a *rank* r *oriented matroid* on the ground set E is any subset $\mathcal{F} \subseteq \{+, 0, -\}^E$ of the form $\mathcal{F} = \mathrm{im}(\sigma_{\mathcal{A}}) \cup \{\hat{0}\}$ for some rank r arrangement of pseudo-spheres $\mathcal{A}$.

Remark 3.2. If we partially order the set of signs $\{+, 0, -\}$ by $0 < +$, $0 < -$ and $+$ incomparable to $-$, the set $\mathcal{F}$ inherits a partial order $\leq_{\mathcal{F}}$ as a subset of the product poset $\{+, 0, -\}^E$. With this partial ordering, $\mathcal{F}$ has a unique minimal element $\hat{0}$ and a set $\mathcal{T}$ of maximal elements, called *topes*.

Notice that, on $\mathcal{F} \setminus \{\hat{0}\}$, the ordering $\leq_{\mathcal{F}}$ coincides with the incidence relation of closed cells of the stratification of S^{r-1}.

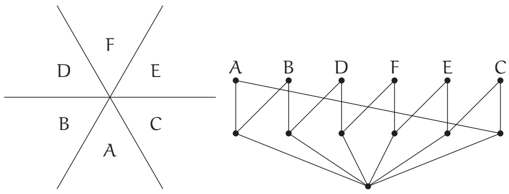

Figure 1. An arrangement of three lines in the real plane, and its poset $\mathcal{F}$ of faces.

Definition 3.3 (Composition of sign vectors)**.** Given two sign vectors $X, Y \in \{+, 0, -\}^E$ define a sign vector $X \circ Y$ as

$$(X \circ Y)_e = \begin{cases} Y_e & \text{if } X_e = 0, \\ X_e & \text{else.} \end{cases}$$

Remark 3.4. If X and Y are covectors of an oriented matroid (and thus correspond to cells on the sphere), then $X \circ Y$ correspond to the cell obtained by 'moving slightly off X towards Y'.

The topological object on which we'll focus is given by the geometric realization of the following poset.

Definition 3.5. The *Salvetti poset* of the given oriented matroid is the set

$$\mathcal{S} = \{(F, C) \in \mathcal{F} \times \mathcal{T} \mid F \circ C = C\}$$

ordered by $(F, C) \le (F', C')$ if $F' \le F$ and $F \circ C' = C$.

Remark 3.6 (Arrangements of hyperplanes). In the particular case where the arrangement $\mathcal{A}$ of Definition 3.1 is induced by the intersection of linear hyperplanes with the unit sphere, Salvetti proved [20] that $|\mathcal{S}|$ can be embedded as a deformation retract into the complement of the complexification of the hyperplanes.

Definition 3.7 (Definition 4.2.9 of [4])**.** Let M be a given oriented matroid with set $\mathcal{F}$ of covectors and set $\mathcal{T}$ of topes. Given $B \in \mathcal{T}$ let $\mathcal{T}_B$ denote the poset of all topes ordered by

$$T \preccurlyeq_B R \Leftrightarrow S(B, T) \subseteq S(B, R)$$

where the *separating set* $S(X, Y)$ of two sign vectors $X, Y \in \{+, 0, -\}^E$ is defined as $S(X, Y) := \{e \in E \mid X_e = -Y_e \neq 0\}$.

Remark 3.8. The *interval* determined by $R \preccurlyeq_B T$ is the subposet of $\mathcal{T}_B$ induced on

$$[R, T] = \{C \in \mathcal{T}_B \mid R \preccurlyeq_B C \preccurlyeq_B T\}.$$

The notation does not reflect the dependency on B because for any choice of B' such that $S(T, R) \cap S(B', T) = \emptyset$ the posets $[R, T] \subseteq \mathcal{T}_B$ and $[R, T] \subseteq \mathcal{T}_{B'}$ are canonically isomorphic (see e.g. [4, Corollary 4.2.11]).

For the purposes of what follows we need to replace $|\mathcal{S}|$ with another, homotopy equivalent simplicial complex.

Definition 3.9. Let $\mathcal{Q} := (\mathcal{T} \times \mathcal{T}, \le)$ be the poset given on the set $\mathcal{T} \times \mathcal{T}$ by the order relation

$$(T, R) \le (T', R') :\Leftrightarrow T \preccurlyeq_{T'} R \preccurlyeq_{T'} R'.$$

We show that $\le$ is transitive, and leave reflexivity and anti-symmetry to the reader. Let $(T, R) \le (T', R')$ and $(T', R') \le (T'', R'')$. Then by definition (a) $T \preccurlyeq_{T'} R \preccurlyeq_{T'} R'$ and (b) $T' \preccurlyeq_{T''} R' \preccurlyeq_{T''} R''$. From (b) it follows in particular $T' \preccurlyeq_{T''} R'$, and the interval $[T', R']$ has, by Remark 3.8, the same structure in $\mathcal{T}_{T''}$ as in $\mathcal{T}_{T'}$. Therefore, from (a) we deduce $T' \preccurlyeq_{T''} T \preccurlyeq_{T''} R \preccurlyeq_{T''} R'$. With (b), this implies $T \preccurlyeq_{T''} R \preccurlyeq_{T''} R''$, meaning $(T, R) \le (T'', R'')$, as required.

Remark 3.10. An oriented matroid M is uniquely determined by its covectors, and also by several other equivalent combinatorial systems, e.g., vectors, basis signatures, or the set of topes. The oriented matroid M is considered to consist of any and all of these notions - see [4]. In particular, the adjacency relation among topes (i.e. the *tope graph* of [4, Definition 4.2.1]) is enough to reconstruct the oriented matroid up to a reorientation (i.e., up to a global change of sign in some components of the covectors). Correspondingly, the poset $\mathcal{Q}$ can be described in terms of the tope graph of M: $(T, R) \le (T', R')$ if and only if some geodesic from T to R can be extended to a geodesic from T' to R'.

Lemma 3.11. *The function*

$$\mathcal{S} \to \mathcal{Q}; \quad (F, C) \mapsto (C, F \circ (-C))$$

is a poset morphism and induces a homotopy equivalence $|\mathcal{S}| \simeq |\mathcal{Q}|$.

Proof. The given function is order-preserving. Indeed, assuming $(F, C) \leq (F', C')$ one sees that $F_e \leq C_e$ implies $F'_e \leq C'_e$ for all $e \in E$. This last statement is equivalent to $(C, F \circ (-C)) \leq_{\mathcal{Q}} (C', F' \circ (-C'))$.

Moreover, for any given $(T, R) \in \mathcal{Q}$ the preimage of $\mathcal{Q}_{\leq (T,R)}$ is

$$\left\{ [F, F \circ T] \middle| \sigma_{\mathcal{A}}^{-1}(F) \in \bigcap_{\substack{e \notin S(T,R), \\ T \in S_e^{\tau}}} S_e^{\tau} \right\}.$$

Here S_e^{τ} denotes S_e^{+} or $S_e^{-} := -S_e^{+}$. This poset is isomorphic to the poset

$$\left\{ F \middle| \sigma_{\mathcal{A}}^{-1}(F) \in \bigcap_{\substack{e \notin S(T,R), \\ T \in S_e^{\tau}}} S_e^{\tau} \right\}^{\mathrm{op}}$$

of those cells in the arrangement of pseudo-spheres that lie in the relative interior of the region containing R and delimited by the pseudo-spheres not separating R from T. This poset is contractible, e.g., by [4, Proposition 4.3.6 (c)] and [3, Theorem 4.1], and we conclude with Remark 2.6. $\square$

Definition 3.12. We define a function $\rho : \mathcal{Q} \to \mathcal{Q}$ by setting $\rho(R, T) := (-T, R)$ for every $(R, T) \in \mathcal{Q}$.

Remark 3.13. The function ρ is evidently a bijection.

Lemma 3.14. *The function ρ is order-reversing (thus, it defines an isomorphism $\mathcal{Q} \simeq \mathcal{Q}^{\mathrm{op}}$). Moreover, $\rho^4 = \mathrm{id}$.*

Remark 3.15. The following technical facts are a corollary of [4, Proposition 4.2.10], and will be used in the proof of Lemma 3.14. For all $A, B, C, D \in \mathcal{T}$:

(a) $A \preccurlyeq_B C \Rightarrow -A \preccurlyeq_{-B} -C$;
(b) $A \preccurlyeq_B C \preccurlyeq_B D \Rightarrow C \preccurlyeq_A D$;
(c) $A \preccurlyeq_B C \Rightarrow B \preccurlyeq_{-C} A$.

Proof of Lemma 3.14. It is enough to prove that ρ is order-reversing, all other claims follow easily. To this end let $(R, T) \leq (R', T') \in \mathcal{Q}$, meaning $R' \preccurlyeq_{R'} R \preccurlyeq_{R'} T \preccurlyeq_{R'} T'$. Now: $R \preccurlyeq_{R'} T$ implies $R' \preccurlyeq_{-T} R$ by Remark 3.15.(c), while from $T \preccurlyeq_{R'} T' \preccurlyeq_{R'} -R'$ we get $T' \preccurlyeq_T -R'$ (Remark 3.15.(b)) and thus $-T' \preccurlyeq_{-T} R'$ (Remark 3.15.(a)). Together, we obtain $-T' \preccurlyeq_{-T} R' \preccurlyeq_{-T} R$, i.e., $(-T, R) \geq (-T', R')$, as required. $\square$

Theorem 3.16. *The assignment $n \mapsto \rho^n$ defines an action of $\mathbb{Z}_4$ on $\Delta(\mathcal{Q})$ (and thus a simplicial action on the complex $|\mathcal{Q}|$).*

Proof. This follows from Lemma 3.14 with Remark 2.4. $\square$

Remark 3.17. The map ρ extends to an order 4 automorphism of $\mathcal{F} \times \mathcal{F}$, but it does not induce an action on the associated 2-matroid [22, Proposition 4.2]. Moreover, even in the realizable case, the quotient of the $s^{(2)}$-stratification associated to the complexification of $\mathcal{A}$ [5, Theorem 5.1(iv)] is not easily identifiable with the decone.

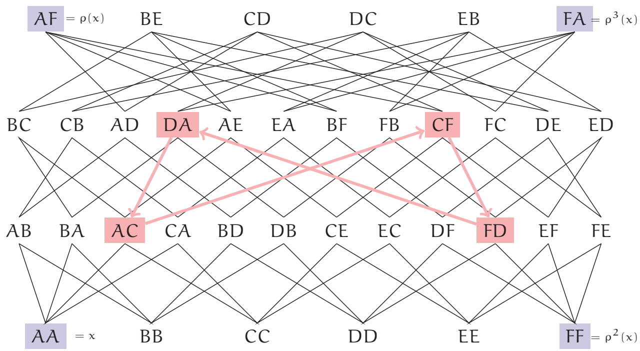

FIGURE 2. The poset $\mathcal{Q}$ for the arrangement of Figure 1. Two orbits of the $\mathbb{Z}_4$-action are shaded. The image of the inclusion of $\mathcal{S}$ given in Lemma 3.11 are all elements of rank $0, 1$ and 3.

4. Combinatorial deconing

Recall that, throughout, $\mathcal{F}$ denotes the poset of covectors of an arbitrary (but fixed) oriented matroid on the ground set E.

Definition 4.1. Every choice of an element $e \in E$ gives rise to an *affine oriented matroid* with poset of covectors

$$d_e\mathcal{F} := \{F \in \mathcal{F} \mid F_e = +\}.$$

From now on an arbitrary element $e \in E$ will be fixed, and we will simply write $d\mathcal{F}$.

Accordingly, we define the subposets

$$d\mathcal{S} := \{[F, C] \in \mathcal{S} \mid F, C \in d\mathcal{F}\} \subseteq \mathcal{S},$$
$$d\mathcal{Q} := \{(R, T) \in \mathcal{Q} \mid R_e = T_e = +\} \subseteq \mathcal{Q}.$$

Remark 4.2. The map of Lemma 3.11 restricts to a poset map $d\mathcal{S} \to d\mathcal{Q}$ which induces homotopy equivalence (because preimages of lower intervals are equal to those with respect to the unrestricted map, thus Remark 2.6 applies).

Definition 4.3. Consider the oriented matroid of rank-one on the ground set $\{e\}$, with sets of covectors and topes $\mathcal{F}_e = \{(+), (0), (-)\}$ and $\mathcal{T}_e = \{(+), (-)\}$. The action of $\mathbb{Z}_4$ on the associated poset $\mathcal{Q}_e = \{(+,+), (+,-), (-,+), (-,-)\}$ is transitive. Choosing $R \in \mathcal{T}_e$ we can identify the elements of $\mathcal{Q}_e$ with elements of $\mathbb{Z}_4$ so that, for $i = 0, \ldots, 3$, $\rho^i(R, R)$ is identified with the class $[i] \in \mathbb{Z}_4$.

Definition 4.4. Define a function $\Psi : \Delta(\mathcal{Q}_e) \times \Delta(d\mathcal{Q})^{op} \to \mathcal{Q}$ so that, for any given chain $\omega = \omega_1 < \cdots < \omega_k$,

$$\Psi(\{[i]\}, \omega) := \begin{cases} \rho^i(\omega_1) & i \text{ even}, \\ \rho^i(\omega_k) & i \text{ odd}, \end{cases}$$
$$\Psi(\{[i], [i+1]\}, \omega) := \Psi([i], \omega) \vee \Psi([i+1], \omega).$$

Remark 4.5 (Joins in $\mathcal{Q}$). Although $\mathcal{Q}$ is certainly not a lattice, the 'join' in the above definition - which should be thought of as 'the minimum among all elements that are above both terms' - is well defined in the cases we need. Indeed, without loss of generality $\Psi([i], \omega) \vee \Psi([i+1], \omega) = (A, B) \vee (-D, C)$ for some $A, B, C, D \in \mathcal{T}$

with $(A, B) \geq (C, D)$, and one sees that the join operation determines the element $(-D, B)$: indeed, $(-D, B)$ is greater than both (A, B) and (C, D) (e.g. by Remark 3.15 (c)) and for every $(R, T) < (-D, B)$, the interval $[R, T] \subseteq \mathcal{T}$, and thus any geodesic from R to T, either does not contain B (hence $(R, T) \not\geq (A, B)$) or it does not contain $-D$ (and then, $(R, T) \not\geq (-D, C)$).

Remark 4.6 (Notation). For ease of notation we will from now on omit all brackets when referring to elements of $\mathcal{Q}_e$ or $\Delta(\mathcal{Q}_e)$, thus writing for instance 12 instead of $\{[1], [2]\} \subset \mathbb{Z}_4$.

Remark 4.7. It will be convenient to examine explicitly the function Ψ. If $\omega = (A_1, B_1) < \ldots < (A_k, B_k)$ is a chain in $d\mathcal{Q}$, we have

$$\begin{array}{ll} \Psi(0, \omega) = (A_1, B_1); & \Psi(01, \omega) = (-B_k, B_1); \\ \Psi(1, \omega) = (-B_k, A_k); & \Psi(12, \omega) = (-A_1, A_k); \\ \Psi(2, \omega) = (-A_1, -B_1); & \Psi(23, \omega) = (B_k, -B_1); \\ \Psi(3, \omega) = (B_k, -A_k); & \Psi(03, \omega) = (A_1, -A_k). \end{array}$$

Lemma 4.8. *The function Ψ defines a poset map and induces a homotopy equivalence.*

Proof. The maps $\Delta(d\mathcal{Q})^{op} \to \mathcal{Q}$ mapping ω to ω_1 and to ω_k are, respectively, order-preserving and order-reversing. It follows that Ψ is order-preserving.

To prove homotopy equivalence, we consider preimages of elements $(C, K) \in \mathcal{Q}$ and verify the condition of Lemma 2.5.

Case 1*:* $C_e = K_e = +$. First, from the explicit description of Ψ in Remark 4.7 notice the poset isomorphism

$$\Psi^{-1}(\mathcal{Q}_{\geq (C,K)}) \cong \{\omega \in \Delta(d\mathcal{Q})^{op} \mid \max \omega \in d\mathcal{Q}_{\geq (C,K)}\}.$$

Given a poset $\mathcal{P}$ with a unique maximal element x, write $\Delta^\dagger(\mathcal{P})$ for the poset of all chains in $\mathcal{P}$ containing x. Define a diagram of posets

$$\mathcal{D} : (d\mathcal{Q}_{\geq (C,K)})^{op} \to \mathrm{Pos}; \quad \mathcal{D}(X, Y) = \Delta^\dagger(\mathcal{Q}_{\leq (X,Y)})^{op}$$

with diagram maps being the natural maps.

Then the Grothendieck construction $\int \mathcal{D}$, viewed as a poset, has elements $((X, Y), \omega)$, where $(X, Y) \geq (C, K)$ and $\max \omega = (X, Y)$, ordered according to

$$((X, Y), \omega) \leq ((X', Y'), \omega') \Leftrightarrow (X, Y) \geq (X', Y'), \text{ and } \omega \supseteq \omega'.$$

Thus we have an evident poset isomorphism $\Psi^{-1}(\mathcal{Q}_{\geq (C,K)}) \cong \int \mathcal{D}$ and so

$$|\Psi^{-1}(\mathcal{Q}_{\geq (C,K)})| \simeq |\textstyle\int \mathcal{D}| \simeq \operatorname{hocolim} |\mathcal{D}|,$$

where the second equivalence is an instance of [21, Theorem 1.2]. Here $|\mathcal{D}|$ is the diagram of geometric realizations of $\mathcal{D}$ in the category of topological spaces and continuous maps.

Now, because all posets $\mathcal{D}(X, Y)$ have a unique minimal element (the 1-element 'chain' $\{(X, Y)\}$), their geometric realizations $|\mathcal{D}(X, Y)|$ are contractible. With [15, Theorem 15.19] we obtain

$$|\Psi^{-1}(\mathcal{Q}_{\geq (C,K)})| \simeq \operatorname{hocolim} |\mathcal{D}| \simeq |d\mathcal{Q}_{\geq (C,K)}| \simeq *.$$

Case 2: $-C_e = K_e = +$. Again, with Remark 4.7 we can write explicitly

$$\begin{aligned}\Psi^{-1}(\mathcal{Q}_{\geq(C,K)}) = &\{(01,(A_1,K') < \ldots < (A_k,-C')) \mid (K',-C') \leq (K,-C)\}\\ &\cup\{(1,(A_1,B_1) < \ldots < (K',-C')) \mid (K',-C') \leq (K,-C)\}\\ &\cup\{(12,(-C',B_1) < \ldots < (K,B_k)) \mid (K',-C') \leq (K,-C)\}\end{aligned}$$

and we call the three parts of the union $\mathcal{P}_{01}$, $\mathcal{P}_1$, $\mathcal{P}_{12}$, in the order listed. It is immediate to see that $\mathcal{P}_1 = \{1\} \times \Delta(\mathcal{Q}_{\leq(K,-C)})^{op}$ and is thus contractible. Moreover, notice that $(1,\omega) \in \mathcal{P}_1$ implies both $(01,\omega) \in \mathcal{P}_{01}$ and $(12,\omega) \in \mathcal{P}_{12}$, for all ω. Thus, by defining $\mathcal{R} := \Delta(\mathcal{Q}_e)_{\geq\{1\}} \times \mathcal{P}_1$, we have a covering of $\Psi^{-1}(C,K)$ by three posets $\mathcal{P}_{01}, \mathcal{R}, \mathcal{P}_{12}$ with $\mathcal{P}_{01} \cap \mathcal{R} \simeq \mathcal{P}_{12} \cap \mathcal{R} \simeq \mathcal{P}_1$ (thus contractible) and $\mathcal{P}_{01} \cap \mathcal{P}_{12} = \emptyset$. By the generalized nerve lemma [15, Theorem 15.24] applied to the covering of $|\Psi^{-1}(C,K)|$ by its subcomplexes $|\mathcal{P}_{01}|$, $|\mathcal{R}|$ and $|\mathcal{P}_{12}|$, the poset $\Psi^{-1}(C,K)$ is contractible if $\mathcal{P}_{01}$ and $\mathcal{P}_{12}$ are.

We are thus left with proving contractibility of $\mathcal{P}_{01}$ (contractibility of $\mathcal{P}_{12}$ follows by a similar argument). To this end, notice first of all that $(A_1,K') < (A_2,B_2) < \ldots < (A_k,-C')$ is a chain if and only if

$$A_k \preccurlyeq_{C'} \ldots \preccurlyeq_{C'} A_1 \preccurlyeq_{C'} K' \preccurlyeq_{C'} B_2 \preccurlyeq_{C'} \ldots \preccurlyeq_{C'} B_{k-1}.$$

We thus obtain an order-reversing bijection

$$\mathcal{P}_I \to \Delta[K',C'] \times \Delta^{\dagger\dagger}[K',-C']; \qquad (01,\omega) \mapsto \omega,$$

where $\Delta^{\dagger\dagger}[K',-C']$ denotes the poset of all chains in $[-C',K']$ containing both $-C'$ and K'. This poset has a unique minimal element $\{-C' \leq K'\}$ and is thus contractible, hence

$$\mathcal{P}_{01} \simeq \Delta[K',C'] \times \Delta^{\dagger\dagger}[K',-C'] \simeq *.$$

The other cases are treated analogously to the above. ☐

Theorem 4.9. *For any oriented matroid* M *and every element* e *of its ground set:*

$$|\mathcal{S}| \simeq S^1 \times |d\mathcal{S}|.$$

Proof. Immediate applying Remark 2.10 to Lemma 4.8. ☐

Corollary 4.10 (Theorem 4.2 of [8])**.** *For oriented matroid* M *and any element* e *of its ground set:* $\pi_1(|\mathcal{S}|) \simeq \mathbb{Z} \times \pi_1(|d\mathcal{S}|)$.

5. Non-realizable groups

We close by exhibiting an oriented matroid M for which the fundamental group $\pi_1(|\mathcal{Q}|) \cong \pi_1(|\mathcal{S}|)$ is not isomorphic to the fundamental group of the complement of any arrangement (complexified or not) of linear hyperplanes in $\mathbb{C}^r$. Thus the homotopy type of $\mathcal{Q}$ is not represented by a complex arrangement complement. To our knowledge no example of either phenomenon has appeared in the literature. The example illustrates that results such as ours extending properties of arrangement groups to the non-realizable case are strict generalizations of the existing theory.

The argument uses the degree-one resonance variety of M, which turns out to be an invariant of the cohomology ring of the group $\pi_1(|\mathcal{Q}|)$. This is a union of linear subspaces of $H^1(\pi_1(|\mathcal{Q}|),\mathbb{C})$ that depends only on the underlying matroid $\underline{M}$ of M, up to linear change of coordinates. The idea is then to reconstruct the rank-one and rank-two flats of the underlying matroid $\underline{M}$ from the linear isomorphism type of its degree-one resonance variety, using its description in terms of multinets (a.k.a. combinatorial pencils). For a particular oriented matroid M of rank-three whose underlying matroid $\underline{\mathsf{M}}$ is not realizable over $\mathbb{C}$, we are able to accomplish this. It follows that the fundamental group $\pi_1(|\mathcal{Q}(\mathsf{M})|)$ cannot arise from any complex hyperplane arrangement, since a generic three-dimensional section of such an arrangement would have the same rank-one and rank-two flats as a non-realizable rank-three matroid. A crucial, delicate step in the argument is to show that the structure of the resonance variety precludes the existence of any non-local components. We refer the reader to [10] for background on Orlik-Solomon algebras, their degree-one resonance varieties, and multinets.

5.1. **Resonance varieties.** Let M be an oriented matroid on ground set E, with associated tope-pair poset $\mathcal{Q}$. Let $A_{\mathbb{Z}} = A_{\mathbb{Z}}(\mathsf{M})$ be the cohomology ring $H^*(|\mathcal{Q}|, \mathbb{Z})$ of $|\mathcal{Q}|$. By [5, 13], $A_{\mathbb{Z}}$ is isomorphic as a graded algebra to the Orlik-Solomon (OS) algebra of the underlying unoriented matroid $\underline{\mathsf{M}}$ of M, the quotient of the exterior algebra on E by the ideal generated by elements of the form $\sum_{k=1}^{p}(-1)^k e_1 \cdots \hat{e_i} \cdots e_p$, where $\{e_1, \ldots, e_p\}$ ranges over the circuits in $\underline{\mathsf{M}}$. We assume $\underline{\mathsf{M}}$ is a simple matroid, which implies $A^1_{\mathbb{Z}} \cong \mathbb{Z}^{\mathsf{E}}$. Moreover, $A_{\mathbb{Z}}$ is generated by $A^1_{\mathbb{Z}}$ and is a free $\mathbb{Z}$-module - see [10].

For a graded algebra $R = \bigoplus_{p\geq 0} R^p$, let $R^{\leq 2} = R/\bigoplus_{p\geq 3} R^p$.

Lemma 5.1. *The graded algebras* $A^{\leq 2}$ *and* $H^{\leq 2}(\pi_1(|\mathcal{Q}|), \mathbb{Z})$ *are isomorphic.*

Proof. By the remarks above, the integral cohomology ring of the space $|\mathcal{Q}|$ is generated in degree-one and is free abelian. Then [18, Proposition 1.6] implies $H^{\leq 2}(|\mathcal{Q}|, \mathbb{Z}) \cong H^{\leq 2}(\pi_1(|\mathcal{Q}|), \mathbb{Z})$. □

Let $A = A_{\mathbb{Z}} \otimes \mathbb{C}$. By the preceding lemma, $A^{\leq 2}$ is determined up to graded algebra isomorphism by $\pi_1(|\mathcal{Q}|)$.

Definition 5.2. The *degree-one resonance variety* of A is the subset $\mathcal{R}^1(A)$ of A^1 given by

$$\mathcal{R}^1(A) = \{a \in A^1 \mid ab = 0 \text{ for some } b \in A^1 - \mathbb{k}a\}.$$

Clearly $\mathcal{R}^1(A)$ depends only on $A^{\leq 2}$. It is not hard to show $\mathcal{R}^1(A)$ is a subset of the diagonal hyperplane $H_0 = \{x \in \mathbb{C}^{\mathsf{E}} \mid \sum_{e\in \mathsf{E}} x_e = 0\}$. Also $\mathcal{R}^1(A)$ is expressible as a finite union of linear spaces of $A^1 \cong \mathbb{C}^{\mathsf{E}}$ [7, 16], every two of which intersect trivially [16]. These maximal linear subspaces of $\mathcal{R}^1(A)$ are called the components of $\mathcal{R}^1(A)$. (In fact they are the irreducible components of $\mathcal{R}^1(A)$, which is an affine algebraic set.)

Corollary 5.3. $\mathcal{R}^1(A)$ *is determined by* $\pi_1(|\mathcal{Q}|)$ *up to linear change of coordinates of* $\mathbb{C}^{\mathsf{E}}$.

Next we review the characterization of components of $\mathcal{R}^1(A)$ from [12]. Each rank-two flat X of cardinality $|\mathsf{X}| \geq 3$ in $\underline{\mathsf{M}}$ gives rise to a component L_{X} of $\mathcal{R}^1(A)$ of dimension $|\mathsf{X}| - 1$, called a *local component*, and defined by

$$L_{\mathsf{X}} = \{x \in H_0 \mid x_e = 0 \text{ for } e \notin \mathsf{X}\}.$$

The *support* $\mathrm{supp}(L)$ of a linear subspace L of A^1 is the set $\{e \in \mathsf{E} \mid x_e \neq 0 \text{ for some } x \in L\}$. The support of a local component L_{X} is the rank-two flat X. A component L of $\mathcal{R}^1(A)$ is *non-local* if $\mathrm{supp}(L)$ has rank greater than two, and is *global* if $\mathrm{supp}(L) = \mathsf{E}$. Non-local components of $\mathcal{R}^1(A)$ arise from multinets supported on rank-three submatroids of $\underline{\mathsf{M}}$, by [12].

Definition 5.4. A (weak) (k,d)-*multinet* on $\underline{M}$ is a pair $(m,\mathcal{N})$ consisting of a function $m\colon E \to \mathbb{N}$ and a partition $\mathcal{N} = E_1 \sqcup \cdots \sqcup E_k$ of E with $k \geq 3$ parts, satisfying

(i) For all i, $\sum_{e\in E_i} m(e) = d$.

(ii) For each rank-two flat $X = \overline{ee'}$ spanned by points e and e' from different parts of $\mathcal{N}$, the sum $\sum_{e\in X\cap E_i} m(e)$ is constant, independent of i.

Multinets are also called combinatorial pencils [17]; they arise from one-dimensional linear systems (pencils) of degree d projective plane curves with k completely reducible (not necessarily reduced) fibers. The set of rank-two flats described in condition (ii) is called the *base locus*, and is denoted $\underline{\mathcal{X}}$. For $X \in \underline{\mathcal{X}}$, the number $\sum_{e\in X\cap E_i} m(e)$ is denoted $m(X)$. A (k,d)-*net* is a (k,d)-multinet satisfying $m(e) = 1 = m(X)$ for all $e \in E$ and $X \in \underline{\mathcal{X}}$. Equivalently, a (k,d)-net is a partition $\mathcal{N}$ of E with parts of size d for which each rank-two flat in $\underline{\mathcal{X}}$ contains one point from each part of $\mathcal{N}$.

In the next sequence of lemmas, we establish some restrictions on the non-local components that can appear, under some restrictions on M.

Lemma 5.5. *Suppose $\underline{M}$ has no rank-two flats of size larger than three. Then any multinet on $\underline{M}$ is a $(3,d)$-net for some d.*

Proof. Suppose $(m,\mathcal{N})$ is a multinet on $\underline{M}$, and $\underline{\mathcal{X}}$ is the associated base locus. Each rank-two flat in $\underline{\mathcal{X}}$ contains at least one point from each block of $\mathcal{N}$, hence $\mathcal{N}$ has 3 blocks and one point from each block is in each flat in $\underline{\mathcal{X}}$. Suppose $m(e) = m > 1$ for some $e \in E$. Without loss, $e \in E_1$. Then every point in E_2 and E_3 must have multiplicity m, which then implies all points in E_1 have multiplicity m, by condition (ii) of Definition 5.4. Then $|E_1| = |E_2| = |E_3|$ by condition (i), hence $\mathcal{N}$ is a net. □

Lemma 5.6. *Suppose $\underline{M}$ is the cycle matroid of a simple graph Γ, and $\mathcal{R}^1(A)$ has a global component. Then Γ is the complete graph K_4.*

Proof. Since $\underline{M}$ is graphic, there are no rank-two flats of size greater than three. Then any non-local component arises from a $(3,d)$-net on $\underline{M}$, by Lemma 5.5. Let us refer to the blocks of the associated partition as colors. Flats in the base locus $\underline{\mathcal{X}}$ are edge sets of triangles (3-cliques) in Γ. Fix one such flat $X = \{e_1, e_2, e_3\}$. Then $E - X$ is non-empty, since $\mathcal{R}^1(A)$ has a non-local component. Any $e \in E - X$ must be the same color as one of e_1, e_2, or e_3, and must lie in a rank-two flat containing the other two. Choose $e_4 \in E - X$. Without loss e_4 has the same color as e_1. Write $\overline{e_2, e_4} = \{e_2, e_4, e_5\}$ and $\overline{e_3, e_4} = \{e_3, e_4, e_6\}$. Any remaining edge of Γ must also lie in a triangle with two of e_1, e_2, or e_3; since $\underline{M}$ is simple no such edge exists. Then $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and Γ is isomorphic to K_4. □

Lemma 5.7. *Suppose $|E| \leq 8$ and $\underline{M}$ has no rank-two flats of size greater than three. If $\mathcal{R}^1(A)$ has a global component, then $\underline{M}$ is the graphic matroid of the complete graph K_4.*

Proof. It is no loss to assume $\underline{M}$ is simple, i.e., $\underline{M}$ has no loops or multiple points. If $\underline{M}$ has a free point, then it cannot support a net. Using the catalog [1] (see also [6]), one finds that only 62 of the 489 rank-three matroids on eight or fewer points are simple, not graphic, and have no free points, and, of these, only 27 have no rank-two flats of size greater than three. One checks by hand that none of these

27 matroids supports a $(3,d)$-net. Combined with Lemma 5.6 and Lemma 5.5, this proves the claim. $\square$

Remark 5.8. In fact Lemma 5.7 holds without the restriction on rank-two flats. For the general result one needs the fact that the partition associated with a multinet on $\underline{M}$ is neighborly, as defined in [11], and then one checks that none of the 62 non-graphic simple rank-three matroids on eight or fewer points with no free points supports a neighborly partition.

Let $\mathcal{C}$ be the set of components of $\mathcal{R}^1(A)$. Let $d_{\mathcal{C}}\colon 2^{\mathcal{C}} - \{\emptyset\} \to \mathbb{Z}_{\geq 0}$ be defined by $d_{\mathcal{C}}(S) = \dim_{\mathbb{C}}(\sum_{L\in S} L)$. The pair $(\mathcal{C}, d_{\mathcal{C}})$ is the *(resonance) polymatroid* of $\underline{M}$, and is denoted by $\mathcal{C}_{\underline{M}}$. We call $d_{\mathcal{C}}(S)$ the *rank* of S. A subset S of $\mathcal{C}_{\underline{M}}$ is *closed* if $d_{\mathcal{C}}(S) < d_{\mathcal{C}}(T)$ for all $T \supsetneq S$. If $S \subseteq \mathcal{C}$, the polymatroid $(S, d_{\mathcal{C}}|_{2^S - \{\emptyset\}})$ is denoted $\mathcal{C}_S$. A polymatroid isomorphism $\varphi\colon (\mathcal{C}, d_{\mathcal{C}}) \to (\mathcal{C}', d_{\mathcal{C}'})$ is a bijection $\varphi\colon \mathcal{C} \to \mathcal{C}'$ satisfying $d_{\mathcal{C}'}(\varphi(S)) = d_{\mathcal{C}}(S)$ for all $S \in 2^{\mathcal{C}} - \{\emptyset\}$. For $S \subseteq \mathcal{C}$ let $\operatorname{supp}(S) = \operatorname{supp}(\sum_{L\in S} L)$. Note that $d_{\mathcal{C}}(S) \leq |\operatorname{supp}(S)| - 1$, since $\sum_{L\in S} L \subseteq \mathbb{C}^{\operatorname{supp}(S)} \cap H_0$.

Let $\underline{M}$ and $\underline{M}'$ be matroids on the same ground set, with Orlik-Solomon algebras A and A'. Let $\mathcal{C} = \mathcal{C}_{\underline{M}}$ and $\mathcal{C}' = \mathcal{C}_{\underline{M}'}$.

Lemma 5.9. *Suppose* $\iota\colon A^1 \to (A')^1$ *is a linear isomorphism carrying* $\mathcal{R}^1(A)$ *to* $\mathcal{R}^1(A')$. *Then* ι *induces a polymatroid isomorphism* $\iota_*\colon \mathcal{C} \to \mathcal{C}'$. *In particular,* S *is a closed subset of* $\mathcal{C}$ *if and only if* $\iota_*(S)$ *is a closed subset of* $\mathcal{C}'$.

Proof. Since ι is a linear isomorphism, it sends components of $\mathcal{R}^1(A)$ to components of $\mathcal{R}^1(A')$, and the induced map $\iota_*\colon \mathcal{C} \to \mathcal{C}'$ is bijective. Also $\dim_{\mathbb{C}}(\sum_{L\in S} L) = \dim_{\mathbb{C}}\left(\sum_{L\in S} \iota(L)\right)$ for any collection of subspaces S of A^1. Thus ι_* is a polymatroid isomorphism. For the last statement, observe that closure is defined in terms of the polymatroid structure. $\square$

We have the following corollary of Lemma 5.7.

Corollary 5.10. *Suppose* $|E| \leq 8$ *and* $\mathcal{R}^1(A)$ *has a non-local component. Then there is a closed subset* S *of* $\mathcal{C}$ *with* $|S| = 5$ *and* $d_{\mathcal{C}}(S) = 5$.

Proof. By [11], the resonance variety of $\underline{M}(K_4)$ has four local components and one global component, and has rank five. If $L \in \mathcal{C}_{\underline{M}}$ is a non-local component, then $\operatorname{supp}(L)$ is a copy of $\underline{M}(K_4)$ in $\underline{M}$, by Lemma 5.7. Then one has a five element subset S of $\mathcal{C}$ with $\operatorname{supp}(S) = \operatorname{supp}(L)$ of size 6 and $d_{\mathcal{C}}(S) = 5$. Moreover S must be closed in $\mathcal{C}_{\underline{M}}$ since $\operatorname{supp}(L)$ cannot support any other net or rank-two flats of size at least three. $\square$

5.2. **Building blocks.** The key to our argument is the existence of some small matroids that are uniquely determined by $\mathcal{C}$. The graphic matroid $M(K_4)$ is one such matroid. Consider now the *rank-three whirl*, the matroid $\underline{W}$ of rank-three on $\{1,2,3,4,5,6\}$ with dependent rank-two flats 123, 345, and 156. The polymatroid $\mathcal{C}_{\underline{W}}$ of $\underline{W}$ has size three and rank five.

Lemma 5.11. *Suppose* $|E| \leq 8$, $S \subseteq \mathcal{C}$ *is a closed subset with* $\mathcal{C}_S$ *isomorphic to* $\mathcal{C}_{\underline{W}}$. *Assume* S *contains no non-local components. Then* $\operatorname{supp}(S)$ *is a six-point submatroid of* $\underline{M}$ *isomorphic to* $\underline{W}$.

Proof. Since S has no non-local components by hypothesis, $\operatorname{supp}(S)$ has at least six points and three rank-two flats X_1, X_2, X_3 of size three. The flats X_1, X_2, and X_3

cannot be pairwise disjoint, else $d_{\mathcal{C}}(S) = 6$. After relabeling, we may assume X_1 and X_2 have a point in common. Then $d_{\mathcal{C}}(\{L_{X_1}, L_{X_2}\}) = 4$. Again because $d_{\mathcal{C}}(S) \neq 6$, X_3 must meet $X_1 \cup X_2$ in two points. Then $|E| = 6$. The only rank-three matroids on six points with three rank-two flats of size three are $\underline{M}(K_4)$ and $\underline{W}$. The resonance variety of $\underline{M}(K_4)$ has no closed sets of size three. Indeed, $\mathcal{C}_{K_4} := \mathcal{C}(\underline{M}(K_4))$ has rank-five, and any two-element subset of $\mathcal{C}_{K_4}$ has rank four, so all five elements of $\mathcal{C}_{K_4}$ lie in the closure of any three-element subset. We conclude that $\mathrm{supp}(S)$ is isomorphic to $\underline{W}$. □

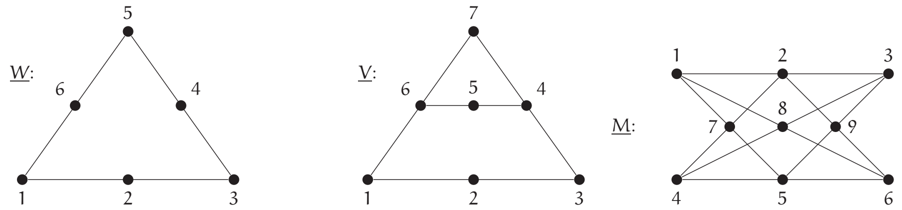

FIGURE 3. The matroids W, V and the non-Pappus matroid M

Let $\underline{V}$ denote the rank-three matroid with ground set $\{1, 2, 3, 4, 5, 6, 7\}$ and dependent rank-two flats 123, 456, 167, and 347. Note that $\underline{V}$ has precisely two deletions isomorphic to $\underline{W}$, $\underline{V} - 2$ and $\underline{V} - 5$. We call the flats 123 and 456 of $\underline{V}$ *distinguished*: they are the unique pair of disjoint rank-two flats. Their complement, the point 7, is the *distinguished point* of $\underline{V}$.

Lemma 5.12. *Suppose* $|E| \leq 8$, $S \subseteq \mathcal{C}$ *is closed with* $|S| = 4$, $d_{\mathcal{C}}(L) = 2$ *for all* $L \in S$, *and* $d_{\mathcal{C}}(S) = 6$. *Suppose* S *has no non-local components, and there are exactly two subsets* $T \subseteq S$ *with* $\mathcal{C}_T$ *isomorphic to* $\mathcal{C}_{\underline{W}}$. *Then*

(i) $\mathrm{supp}(S)$ *is a seven-point submatroid of* $\underline{M}$ *isomorphic to* $\underline{V}$*;*
(ii) *the distinguished rank-two flats of* $\mathrm{supp}(S)$ *are the supports of the unique pair of elements of* S *that do not lie in a single copy of* $\mathcal{C}_{\underline{W}}$*; and*
(iii) *the distinguished point of* $\underline{V}$ *is the unique point in the support of the other two elements of* S.

Proof. Let $T \subseteq S$ with $\mathcal{C}_T$ isomorphic to $\mathcal{C}_{\underline{W}}$. By Lemma 5.11, $\mathrm{supp}(T)$ is a six-point submatroid of $\underline{M}$ isomorphic to $\underline{W}$. Write $S = T \cup \{L\}$. Since $d_{\mathcal{C}}(S) = d_{\mathcal{C}}(T) + 1$, $|\mathrm{supp}(S)| \leq |\mathrm{supp}((T)| + 1 = 7$. Since T is closed and $d_{\mathcal{C}}(T) = 5 = |\mathrm{supp}(T)| - 1$, $\mathrm{supp}(L) \not\subseteq \mathrm{supp}(T)$. Then $|\mathrm{supp}(S)| = 7$. Then the submatroid of $\underline{M}$ on $\mathrm{supp}(S)$ is isomorphic to a one-point extension of $\underline{W}$. Since L is a local component, this extension has four three-point lines. There are, up to isomorphism, two such extensions, and, of these, $\underline{V}$ is distinguished by the fact that it has two deletions isomorphic to $\underline{W}$; the other has three. Then the hypothesis of the lemma implies $\mathrm{supp}(S)$ is isomorphic to $\underline{V}$. The latter two statements are easily verified. □

5.3. **The main example.** Let M be the oriented matroid of the non-Pappus arrangement of pseudo-lines [14, Theorem 3.2 and Figure 3.3]. The underlying rank-three matroid $\underline{M}$ has nine points, which we identify with the numbers $1, \ldots, 9$, and eight dependent rank-two flats

$$123, 157, 168, 247, 269, 348, 359, 456.$$

In any point configuration over a field with these dependent rank-two flats, 789 is also a rank-two flat, by Pappus' Theorem. Thus $\underline{M}$, the non-Pappus matroid, is not realizable over any field. Then the oriented matroid M is non-realizable. Let $\mathcal{Q}$ be the tope-pairs poset associated with M.

Theorem 5.13. $\pi_1(|\mathcal{Q}|)$ *is not isomorphic to the fundamental group of the complement of any arrangement of linear hyperplanes in* $\mathbb{C}^r$.

Proof. Suppose $\mathcal{A}$ is an arrangement of linear hyperplanes in $\mathbb{C}^r$, and $\pi_1(|\mathcal{Q}|)$ is isomorphic to the fundamental group of the complement of $\mathcal{A}$. Let $\underline{M}'$ denote the underlying matroid, and A' the Orlik-Solomon algebra of $\mathcal{A}$. By Corollary 5.3 there is a linear isomorphism from $\iota\colon A^1 \to (A')^1$, with $\iota(\mathcal{R}^1(A)) = \mathcal{R}^1(A')$, and ι induces an isomorphism $\iota_*\colon \mathcal{C} \to \mathcal{C}'$, by Lemma 5.9. The non-Pappus matroid $\underline{M}$ has no rank-two flats of size greater than three, it does not support a net, and it has no submatroids isomorphic to $\underline{M}(K_4)$. Then, by [12, Corollary 3.12] and Lemmas 5.5 and 5.7, $\mathcal{R}^1(A)$ consists of eight two-dimensional local components. For $L \in \mathcal{C}$, denote $\iota_*(L) \in \mathcal{C}'$ by L'.

First, since $(A')^1 \cong A^1$, $\underline{M}'$ is a matroid on nine points. Moreover, since ι is an isomorphism, $\mathcal{R}^1(A')$ consists of eight two-dimensional subspaces. Then $\underline{M}'$ has at most eight dependent rank-two flats, and none of size larger than three.

Then, by [12] and Lemma 5.5, any global component in $\mathcal{C}'$ must be supported by a net, while any net on nine points has at least nine dependent flats in its base locus. Thus $\mathcal{C}'$ has no global components.

A *Mathematica* computation shows that the only sets $S \subseteq \mathcal{C}$ satisfying $d_{\mathcal{C}}(S) = 5$ are closed and consist of three components. There are 12 of them, corresponding to the 12 copies of $\underline{W}$ in $\underline{M}$:

$$\{L_{123}, L_{157}, L_{168}\},\ \{L_{123}, L_{157}, L_{359}\},\ \{L_{123}, L_{168}, L_{269}\},\ \{L_{123}, L_{157}, L_{348}\},$$

$$\{L_{123}, L_{247}, L_{348}\},\ \{L_{123}, L_{269}, L_{359}\},\ \{L_{157}, L_{168}, L_{456}\},\ \{L_{157}, L_{247}, L_{456}\},$$

$$\{L_{168}, L_{348}, L_{456}\},\ \{L_{247}, L_{269}, L_{456}\},\ \{L_{269}, L_{359}, L_{456}\}, \{L_{348}, L_{359}, L_{456}\}.$$

The images of these sets are precisely the subsets S' of $\mathcal{C}'$ satisfying $d'_{\mathcal{C}}(S') = 5$, and they are closed, by Lemma 5.9, and have only three elements. Then $\mathcal{C}'$ has no non-local components, by Corollary 5.10.

The elements L_{123} and L_{456} of $\mathcal{C}$ are distinguished by the fact that they are included in four copies of $\mathcal{C}_W$ in $\mathcal{C}$. All other elements of $\mathcal{C}$ lie in six copies of $\mathcal{C}_{\underline{W}}$. Moreover, L_{123} and L_{456} lie in precisely three copies of $\mathcal{C}_{\underline{V}}$, with supports 1234567, 1234568, and 1234569. The lines 123 and 456 are the distinguished rank-two flats in each of the three. We label these copies of $\mathcal{C}_{\underline{V}}$ by their distinguished points, as $\mathcal{C}_7$, $\mathcal{C}_8$, and $\mathcal{C}_9$.

Let $\mathcal{C}'_i$ denote the image of $\mathcal{C}_i$ in $\mathcal{C}'$, for $i = 7, 8, 9$. By Lemma 5.12, $\operatorname{supp}(\mathcal{C}'_7)$ is a copy of $\underline{V}$ with distinguished lines given by the supports of L'_{123} and L'_{456}. We label the elements of these supports $1', 2', 3'$ and $4', 5', 6'$, respectively. Moreover, the distinguished point in $\operatorname{supp}(\mathcal{C}'_7)$ is the unique point in $\operatorname{supp}(L'_{157}) \cap \operatorname{supp}(L'_{247})$; we label it $7'$. Similarly, the intersections $\operatorname{supp}(L'_{168}) \cap \operatorname{supp}(L'_{348})$ and $\operatorname{supp}(L'_{269}) \cap \operatorname{supp}(L'_{359})$ each consists of a single point – the distinguished points of $\operatorname{supp}(\underline{V}'_8)$ and $\operatorname{supp}(\underline{V}'_9)$, respectively, which we label $8'$ and $9'$.

The three copies of $\underline{V}$ yield eight rank-two flats of size three in $\underline{M}'$, and with our labeling they coincide with the dependent rank-two flats of $\underline{M}$. Thus the truncation of $\underline{\mathsf{M}}'$ to rank-three is isomorphic to $\underline{\mathsf{M}}$. This is a contradiction: a generic three-dimensional section of $\mathcal{A}'$ would be a realization of $\underline{\mathsf{M}}$, which is not realizable over $\mathbb{C}$. $\square$

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